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Remarks on countability and star covering properties

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ABSTRACT

In this paper, we construct an example of a T_4 feebly Lindelöf space X which is not star Lindelöf under $2^{\aleph_0} = 2^{\aleph_1}$, which gives a partial answer to Alas, Junqueira and Wilson (2011) [1, Question 4].

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1. Introduction

By a space, we mean a topological space. The purpose of this paper is to give an example stated in the abstract. In the rest of this section, we give definitions of terms which are used in the example. Let X be a space and \mathcal{U} a collection of subsets of X . For $A \subseteq X$, let $St(A, \mathcal{U}) = \bigcup \{U \in \mathcal{U} : U \cap A \neq \emptyset\}$.

Definition. ([1,5]) Let P be a topological property. A space X is said to be *star P* if whenever \mathcal{U} is an open cover of X , there exists a subspace $A \subseteq X$ with property P such that $X = St(A, \mathcal{U})$. The set A will be called a *star kernel* of the cover \mathcal{U} .

The term *star P* was coined in [1,5] but certain star properties, specifically those corresponding to “ \mathcal{P} = finite” and “ \mathcal{P} = countable” were first studied by van Douwen et al. in [2] and later by many other authors. A survey of star covering properties with a comprehensive bibliography can be found in [4]. Here, we use the terminology from [1,5]. In [4] and earlier [2], a star finite space is called *starcompact* and *strongly 1-starcompact*, and a star countable space is called *star Lindelöf* and *strongly 1-star Lindelöf*. In [6], a star σ -compact space is called *σ -starcompact*. In [1], Alas, Junqueira and Wilson studied the relationships of star P properties for $P \in \{\text{Lindelöf}, \sigma\text{-compact}, \text{countable}\}$ with other Lindelöf type properties. Recall from [4] that a space X is *feebly Lindelöf* if every locally finite family of non-empty open sets in X is countable. In [1], Alas, Junqueira and Wilson showed that a star Lindelöf space is feebly Lindelöf, and asked the following question:

Question. ([1]) Is a T_4 feebly Lindelöf space star Lindelöf?

The purpose of this note is to construct an example of a T_4 feebly Lindelöf space X that is not star Lindelöf under $2^{\aleph_0} = 2^{\aleph_1}$, which gives a partial answer to the above question.

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Throughout this paper, the cardinality of a set A is denoted by $|A|$. Let ω denote the first infinite cardinal and ω_1 the first uncountable cardinal. As usual, a cardinal is the initial ordinal and an ordinal is the set of smaller ordinals. For each ordinals α, β with $\alpha < \beta$, we write $(\alpha, \beta) = \{\gamma: \alpha < \gamma < \beta\}$. Every cardinal is often viewed as a space with the usual order topology. Other terms and symbols follow [3].

2. An example of a T_4 feebly Lindelöf space that is not star Lindelöf

In this section, we construct an example stated in the abstract by using the well-known example. But we include the original construction here for the sake of completeness. Recall that a family \mathcal{U} of subsets of the set X is *independent* if for any collection $U_1, U_2, \dots, U_n, V_1, V_2, \dots, V_m$ of distinct elements of \mathcal{U} , $(\bigcap_{i \leq n} U_i) \cap (\bigcap_{j \leq m} (X \setminus V_j)) \neq \emptyset$.

Example 2.1. ([7, Example E]) Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists a separable normal T_1 space with an uncountable discrete closed subspace.

Construction. Let L be a set of cardinality \aleph_1 disjoint from ω . Let $\{F_\alpha: \alpha < \mathfrak{c}\}$ be an independent family of subsets of ω . Using $2^{\aleph_0} = 2^{\aleph_1}$, we may construct a complement-preserving map f from $\wp(L)$ onto $\{F_\alpha: \alpha < \mathfrak{c}\} \cup \{\omega \setminus F_\alpha: \alpha < \mathfrak{c}\}$. Let $X = L \cup \omega$ be with a subbase φ for a topology defined by

- (1) if $M \subseteq L$, then $M \cup f(M) \in \varphi$;
- (2) if $n \in \omega$, then $\{n\} \in \varphi$;
- (3) if $p \in X$, then $X \setminus \{p\} \in \varphi$.

The space X is a separable normal T_1 space with an uncountable discrete closed subspace.

Example 2.2. Assuming $2^{\aleph_0} = 2^{\aleph_1}$, there exists a T_4 feebly Lindelöf space $S(X, \omega)$ which is not star Lindelöf.

Proof. Let $X = L \cup \omega$ be the same as in the construction of Example 2.1. Let

$$S(X, \omega) = L \cup (\omega_1 \times \omega)$$

and topologize $S(X, \omega)$ as follows: A basic neighborhood of $l \in L$ in $S(X, \omega)$ is a set of the form

$$G_{U, \alpha}(l) = (U \cap L) \cup ((\alpha, \omega_1) \times (U \cap \omega))$$

for a neighborhood U of l in X and $\alpha < \omega_1$, and a basic neighborhood of $(\alpha, n) \in \omega_1 \times \omega$ in $S(X, \omega)$ is a set of the form

$$G_V((\alpha, n)) = V \times \{n\},$$

where V is a neighborhood of α in ω_1 . Since $S(X, \omega)$ has a dense subspace $\omega_1 \times \omega$ which is a countable union of pseudo-compact spaces, then $S(X, \omega)$ is feebly Lindelöf.

Next, we show that $S(X, \omega)$ is not star Lindelöf. Since $|L| = \aleph_1$, we can enumerate L as $\{l_\alpha: \alpha < \omega_1\}$. Since $\{l_\alpha: \alpha < \omega_1\}$ is discrete closed in X , then, for each $\alpha < \omega$, there exists an open neighborhood V_α of l_α in X such that

$$V_\alpha \cap L = \{l_\alpha\}.$$

Let us consider the open cover

$$\mathcal{U} = \{G_{V_\alpha, \alpha}(l_\alpha): \alpha < \omega_1\} \cup \{\omega_1 \times \omega\}$$

of $S(X, \omega)$. It remains to show that $St(B, \mathcal{U}) \neq S(X, \omega)$ for every Lindelöf subset B of $S(X, \omega)$. To show this, let B be a Lindelöf subset of $S(X, \omega)$. Since $B \cap L$ is countable, there exists $\beta' < \omega_1$ such that

$$B \cap \{l_\alpha: \alpha > \beta'\} = \emptyset.$$

On the other hand, for each $n \in \omega$, there exists an $\alpha_n < \omega_1$ such that

$$B \cap ((\alpha_n, \omega_1) \times \{n\}) = \emptyset,$$

since B is Lindelöf. Let $\beta'' = \sup\{\alpha_n: n \in \omega\}$, then $\beta'' < \omega_1$. If we pick $\beta_0 > \max\{\beta', \beta''\}$, then

$$l_{\beta_0} \notin St(B, \mathcal{U}),$$

since $G_{V_{\beta_0, \beta_0}}(l_{\beta_0})$ is the only element of \mathcal{U} containing l_{β_0} and $G_{V_{\beta_0, \beta_0}}(l_{\beta_0}) \cap B = \emptyset$, which shows that $S(X, \omega)$ is not star Lindelöf.

Finally, we show that $S(X, \omega)$ is normal. Let E and F be disjoint closed subsets of $S(X, \omega)$. Let us put

$$E_L = E \cap L \quad \text{and} \quad F_L = F \cap L$$

and

$$E_n = E \cap (\omega_1 \times \{n\}) \quad \text{and} \quad F_n = F \cap (\omega_1 \times \{n\}) \quad \text{for each } n \in \omega.$$

Since $\omega_1 \times \{n\}$ is homeomorphic to ω_1 , there exist disjoint clopen subsets E'_n and F'_n in $\omega_1 \times \{n\}$ such that

$$E_n \subseteq E'_n \quad \text{and} \quad F_n \subseteq F'_n \quad \text{for each } n \in \omega.$$

Since E_n and F_n cannot both be cofinal in $\omega_1 \times \{n\}$, thus we may choose E'_n and F'_n in the following way

$$E'_n \text{ is cofinal in } \omega_1 \times \{n\} \quad \text{if and only if} \quad E_n \text{ is cofinal in } \omega_1 \times \{n\} \quad (1)$$

and

$$F'_n \text{ is cofinal in } \omega_1 \times \{n\} \quad \text{if and only if} \quad F_n \text{ is cofinal in } \omega_1 \times \{n\} \quad (2)$$

for each $n \in \omega$. Let

$$\bar{E} = E_L \cup \bigcup_{n \in \omega} E'_n \quad \text{and} \quad \bar{F} = F_L \cup \bigcup_{n \in \omega} F'_n.$$

Then

$$E \subseteq \bar{E}, \quad F \subseteq \bar{F} \quad \text{and} \quad \bar{E} \cap \bar{F} = \emptyset.$$

It follows from (1) and (2) and the construction of the topology of $S(X, \omega)$ that \bar{E} and \bar{F} are closed in $S(X, \omega)$. Without loss of generality, we can assume that $E = \bar{E}$ and $F = \bar{F}$. Since E_L and F_L are disjoint closed subsets of X , then there exist disjoint open subsets U_E and U_F in X such that

$$E_L \subseteq U_E \quad \text{and} \quad F_L \subseteq U_F.$$

Let

$$V_E = (U_E \cap E) \cup \bigcup_{n \in U_E \cap \omega} (\omega_1 \times \{n\}) \quad \text{and} \quad V_F = (U_F \cap F) \cup \bigcup_{n \in U_F \cap \omega} (\omega_1 \times \{n\}).$$

Then, V_E and V_F are disjoint open subsets in $S(X, \omega)$ and $E_L \subseteq V_E$ and $F_L \subseteq V_F$. If we put

$$W_E = E \cup (V_E \setminus F) \quad \text{and} \quad W_F = F \cup (V_F \setminus E).$$

Then, W_E and W_F are disjoint open subsets in $S(X, \omega)$ such that

$$E \subseteq W_E \quad \text{and} \quad F \subseteq W_F,$$

which completes the proof. \square

Remark. It is well-known that $2^{\aleph_0} = 2^{\aleph_1}$ implies negation of CH. Thus, Example 2.2 gives a partial answer to the above question. But the author does not know if there exists a ZFC counter example to the question.

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